Numbers and Maps: The Dynamic Interaction of Internal Meanings and External Resources in Use

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We describe the work of two groups of six 8 year-old children as they plan Father Christmas's epic journey on December 24th. We trace how the children draw upon formal and personal knowledge to make connections with a range of external representations. We conclude that the context, despite its distracting potential, is critical in supporting engagement, while the representations act in mutual support of the construction of a *utility* for directed number (Ainley & Pratt, 2002).

Introduction

Mathematics in text books and indeed in conventional classrooms is often presented as exercises or worksheets in which the mathematics itself has been processed into a form that is easily digested. This *McDonald's* version of mathematics ensures that the mathematical skill or technique is laid bare and typically the sole focus of attention. In this paper the mathematical focus is directed number though the reader will soon become aware that the children's activity spans a rich panoply of disciplines. By *directed number*, we refer to positive and negative numbers. In this context, McDonald's mathematics might take the form of an exercise in which the children are presented with a series of additions of one negative number to another.

Ainley and Pratt (2002) have argued that, as a result, mathematics learning often becomes sterile. Children gain no sense of the bigger picture and activity is not driven by the task itself. In contrast Ainley and Pratt wish to promote tasks in which children do construct a *purpose* for the activity. Ainley and Pratt propose the construct of *utility* of a mathematical concept. They argue that a crucial aspect of a concept that is often given insufficient attention is how that concept might be useful. The McDonald's approach is unlikely to generate utility since the skills are intentionally isolated from any context. Ainley and Pratt see the teacher's planning problem as one of constructing tasks that are likely to be both purposeful and yet lead to the construction of intended utilities.

However, when children work on tasks that encompass meaningful contexts, the mathematical ideas have necessarily to sit alongside a whole panoply of knowledge and ideas drawn from everyday experience and knowledge from other disciplines. We recognise that such knowledge is double-edged. It may support the learning or it may act as a distraction. Students may *overgeneralise* either on the basis of everyday knowledge brought into the mathematical problem, or because they apply rules blindly, having failed to make those rules truly meaningful (Ben-Zeev, 1996; VanLehn, 1986).

The following advice, which seems to go some way towards advocating the McDonald's approach to mathematics teachers, is taken from the Department for Education and Skills TeacherNet website (2004):

> The structured nature of mathematical knowledge suits a structured teaching style. Break down content into relatively small chunks and ensure that students have fully mastered each one before going on to the next step. This will build students confidence about their ability… Children easily develop misconceptions about the meaning of mathematical concepts. Primary school pupils will often acquire a rule and then overgeneralise it to situations in which it is not applicable.

But we question how students can possibly learn what is salient when they are confronted with a diet of McDonald's mathematics. It is our premise that only by embracing context can children begin to discriminate between the salient and the irrelevant. Nevertheless, we must embrace context in an artful way if the purpose of the task is to lead to mathematical utility.

Method

Our aspiration is to trace the use of internal knowledge, whether of a formal mathematical nature or seemingly drawn idiosyncratically from everyday experience, and to relate the use of that knowledge to a range of external representations, intended to support the construction of a utility for directed number. We will use the term *resources* to blur the distinction between internal and external structures or representations. *Webbing* (Noss & Hoyles, 1996) is a term used to describe the process of forging and reforging connections between internal and external resources in use. By bringing external and internal resources into the same framework, webbing recognises the deep relationship between knowledge and setting (Lave, 1988). One construct used in the webbing framework is that of *situated abstraction*. When children try to make sense of behaviour within a setting, they abstract heuristics about that behaviour. Such heuristics are articulated in terms of the structuring resources within that setting. Although they may not at the time of construction have any relevance outside of that setting, they remain as resources which, under certain circumstances, have the potential to be used in novel contexts.

Our approach is to examine the activity of children through the lens of webbing as they draw upon a range of internal and external resources during purposeful activity. We worked for about an hour with each of two separate groups of 6 eight year-old children. The researcher, labelled M in the protocols below, was an experienced teacher, who used herself as a resource for the children. In particular the researcher, in her teaching role, structured the activity, enabling collaborative and disciplined activity. The activity was videotaped. The tapes were then subjected to a process of progressive focussing (Robson, 1993). Initially the videotape was turned into plain discursive accounts, each minute translated into a description of what activity took place during that time. The accounts were examined by both authors of this paper and some of the account was placed to one side as being unconnected with the focus of this study. At the same time, it was clear that the remaining tape contained four phases. The researcher then returned to the tape and transcribed those four phases. Examination of the transcripts revealed a range of internal and external resources and these resources were coded. In the next rendering, the codes were superimposed upon the transcripts. Both authors then discussed the patterns apparent in the use of resources across the four phases. We formulated conjectures for how the children appeared to be using those resources and tested those conjectures across both groups. In this paper, we call upon those comparisons in an attempt to characterise how connections were forged and reforged by the children. In the next subsection, we describe the task that the children were set.

Father Christmas's Epic Journey

The children were told that they were going to plan Father Christmas's journey on Christmas Eve as he delivered presents to various countries. We will describe the task in terms of the four phases identified during the analysis. In the first phase of the task, the

children could use a Father Christmas character to trigger information on a computer about different countries[1]. The computer screen showed a country, its average temperature on Christmas Eve and a picture of Father Christmas wearing clothes appropriate to that temperature (Figure 1). The children wrote that temperature onto a matching card. These temperatures ranged from Nigeria, 27°C to Svalbaard, -30°C. Later the children were asked to order the cards from hottest to coldest in order to create Father Christmas's journey. In the second phase of the task, the children were asked to locate the countries on a map that was placed over strips in such a way that each strip fell below the corresponding country (Figure 2). In the third phase, the children were asked to create an alternative journey for Father Christmas by reference to the map. In the final phase, the children were asked to write appropriate temperatures onto the cards for the countries chosen for the alternative journey. It is worth noting that the children were unable to use the computer to trigger this information since the technology was designed only to work for the pre-planned countries. Thus, the children were required to create directed numbers rather than use those generated on the computer.

Figure 1. The screen picture.

Figure 2. The map.

Illustrating the Webbing of Internal and External Resources

We now wish to illustrate the webbing process through a number of characteristic excerpts from the children's activity.

Introducing the Resources

As the children matched the strips to the cards and ordered them, they used not only their formal knowledge about numbers to carry out this activity, but also a variety of internal and external resources to help them reach their judgments.

For example we see (minute 12) how Cliona (Cl) was using the information on the cards about Father Christmas's clothes to make sense of the temperature information:

1. *Fiona places Father Christmas on a strip and the display shows Nigeria*
2. Cl It's not very hot for just wearing pants.
3. M 27 degrees is really hot
4. S It is.

Children also brought to this activity significant personal knowledge and experience about countries they had visited. For example, David (D) and Liam (L) shared their holiday experiences when discussing Greece (minute 13):

5. D It's hot because it's near Cyprus, isn't it?
6. M Well done, it's very close to Cyprus
7. L And near to Turkey

Also, the children already had some knowledge about temperatures which they used to help them to assimilate the new information. Here four children are discussing the temperature in Poland (minute 15):

8. *D* Zero wow!
9. *F* Zero?
10. *S* Zero! How cold is that?
11. *Cl* Freezing!

In this first phase, the children were simply making sense of the directed numbers representing the temperatures of the countries. In the next phase they began to use this information.

Locating the Journey

The map was now available to the children: it had been laid over the reader strips and the small model of Father Christmas triggered a screen display through the map when placed on one of the countries along the initial route from Nigeria to Svalbaard. The children used their knowledge of numbers to locate the countries in as much as their knowledge of numbers helped them know whereabouts on the map they should look for a country in relation to the other countries along the route. They did not, however, only use their knowledge of numbers: knowledge about countries, maps and temperatures as well as other personal knowledge was also drawn upon. At one point (minute 28), Cliona is trying to locate Hungary:

12. M Is it going to be this side of Father Christmas or the other side of Father Christmas…?
13. Cl There.
14. M Yeh. Why do you think you should be looking in this part?
15. Cl Because it's going up
16. M And that means the temperature's …?
17. Cl Colder

Support from the teacher helps Cliona to locate the country. Cliona also needed the map, her knowledge of the temperature of Hungary and Greece and knowledge about numbers.

Although their sense of numbers helped the children to create the journey, it is interesting to note how this internal resource was webbed with the external resources. The map itself was used almost constantly throughout this phase of the activity, sometimes by only one or two children at a time, often by most (even all) of the group. The map was used to support communication (minute 26):

18. M Do you know where Greece is?
19. S Yes
20. M Can you see where you're going?
21. F They just keep going …. *(she points at the map 3 or 4 times in progressively northward direction)*

Fiona's gesticulation at the map (line 21) is used to show the sense of the journey.
Generally this type of reference to the map helped the children to share information, test ideas, and formulate questions and answers.

Webbing also consisted of connections with the cards, though only to a limited extent in this phase. One child, Marie, had somewhat idiosyncratically taken ownership of the

cards, laying them out in front of her on the table in order. She interacted with the map much less than the other children, but was quick to supply information about the countries and temperatures and became a resource for the group.

We mention this not only to emphasise the complex interrelationship between internal knowledge of numbers and the external resources in the form of the map, screen and cards, but also to highlight how different children create their own personal web. Marie's use of the cards was quite different from others in the group but she nevertheless performed an important role in the collaboration. Indeed the webbing was even more complex than this. Internal resources were not limited to numbers. In locating countries, children called upon personal knowledge of those countries and recently acquired knowledge about latitude.

Our evidence illustrates most effectively how children call upon a diverse and complex range of internal and external resources in locating the journey. We also see how the type and blend of resources chosen by each child might vary.

Creating a Journey

In this phase, children created a new journey using their new knowledge about maps. They did not concern themselves with the numbers but with the relationship between latitude and temperature. The challenge set by the researcher-as-teacher was "to work out a return route from the North Pole to somewhere hot enough that he just needs his swimming trunks on".

These children were able to use prior knowledge about features of maps and globes in this phase. We saw internal and external resources being used to support each other in several ways. We are permitted some insight into David learning about the link between latitude and temperature (minute 36) when Lily (L) chose Saudi Arabia as one stop in Father Christmas's journey:

22. L Saudi Arabia
23. D That's not. Is it?
24. M Well, do you think it might be warmer or colder?
25. D Warmer
26. M Why do you think it's going to be warmer?
27. D Because ….
28. M I think you're right but why do you think so?
29. S Because its farther down from the North Pole
30. D Yeah.

In this excerpt David seemed to be starting to understand the connection between south and warmer but he still needed the help of his most able friend Simon (S). Later David showed more mastery of this idea when he tried to choose a country after (and hotter than) Somalia (minute 39):

31. D Then you could go to Kenya
32. F And I'm going to Kenya
33. D Can we finish in Kenya?
34. M Is Kenya hotter than Somalia?
35. D It goes to the side though
36. M Yes ..?
37. D You could go to Tasmania *(D actually means Tanzania)*

David used the map to see which countries satisfy the conditions he was looking for. He knew that he must find a country which was further south than Somalia. Kenya bordered Somalia south and west and, though David thought that it was generally south and therefore likely to be hotter than Somalia, he was not completely confident in this

choice because parts of Kenya "go to the side" of Somalia. He was more comfortable with a country which was wholly south of Somalia and so suggested Tanzania. By webbing the external resources of the map, the teacher's and Simon's support, he re-used in more independent ways his situated abstraction that "farther from the North Pole means hotter".

Estimating Temperatures

In this fourth phase, the children were challenged to propose temperatures for the countries in their new journey. This was a mathematically rich phase, marked by the children's use of their knowledge of the map to interpolate and extrapolate directed numbers.

In minute 47, Carl (C), Cliona and Lily work together to extrapolate a temperature for Ukraine, based on their knowledge of the temperatures of Poland and Hungary.

38. C Poland was zero and the next one's Hungary
39. M There's Hungary ..
40. Cl … which was 2
41. M 2. So what do we think about Ukraine?
42. L I think it'll be one.

There were also many examples of interpolation. In trying to estimate the temperature of Russia, the Carl, Lily and Simon are able to interpolate with negative numbers (minute 46).

43. M Can you remember what Finland and Poland are?
44. L I know Finland … um minus 12
45. C Minus 12 yeah, so
46. M Finland is minus 12 and Poland is zero so what do you think Russia might be?
47. S Ooh 5
48. M About 5?
49. Ma 6
50. M Yeh, I think 5 or 6 - something in between zero and minus 12
51. L I think it's 5 or 6
52. M Is it 6 or minus 6?
53. S Minus 5 or 6
54. C Minus

In each of these examples, the children were using the map, alongside their knowledge of the temperatures of adjacent countries, to estimate the temperature of an intermediate country. However, as usual in stories about webbing processes, the children were able to call upon a range of resources. Sense-making activity is punctuated by the triggering of unexpected and sometimes unhelpful internal resources, often called misconceptions, though we prefer the term naïve resources (cf Smith et al, 1993).

An interesting example began (minute 37) when the teacher asked them to estimate the temperature for Spain. Spain is a very popular place for families to take their summer holidays, so several of the children think of Spain as very hot.

55. A It is, it is … very hot
56. L About 15?
57. L 15, yeh 15
58. E 15 or 16
59. M Right, so do we know anything about any of the countries?
60. E Portugal's hot, Spain's hot and France is hot and they are all next door neighbours so Spain's.

They used the map but only to confirm what they knew about Spain, forgetting that this epic journey was to take place in winter. Only through the teacher's efforts to steer the children towards using the latitudes on the map as the relevant resource rather than their summer holidays, were the children able to recognize an apparent contradiction. Another example of how webbing can involve the use of incorrect but established internal resources took place in minute 29:

61. J Cos that's bigger so it's going to have more area in it
62. E So probably .. colder
63. M You think the bigger countries do what?
64. J are going to be colder because more air is going to spread around cos like when you've had water in something
65. A Yeh, but look at England
66. J It goes really cold after it's hot
67. A Yeh but James, England is colder than Spain
68. E Oh yeh … and that's really small isn't it?
69. A Yeh
70. E And that's a lot colder than Chad
71. A Yeh
72. M So does that mean that James that your sort of logical thinking there has got something wrong with it
73. J I think I'm wrong

We do not know the source of James's (J) abstraction that "larger countries are colder", but the activity with the map and feedback from his peers, allowed James to question this piece of knowledge.

We have seen in the final phase, how the children forged connections between the map and their knowledge of temperatures to interpolate and extrapolate new knowledge, encountering along the way other internal resources, which needed to be accounted for in the sense-making activity.

Discussion

Of course a study such as this must be tentative in its claims. We are not able to infer statistical generalisations on the basis of work with two groups of students but the account nevertheless illuminates the webbing process. Observation of the task, in which children created journeys for Father Christmas, took place across four phases. Analysis of the children's activity in these four phases showed changes in patterns, starting from a position in which the children worked with numbers and webbed this knowledge with a range of external resources, in particular a map that was provided. We might schematise this webbing as a function: (map, number)→journey. The map is an external resource and the numbers are internal. The entity created, the journey, seemed to coexist both mentally and physically. This function perhaps characterises webbing as experienced by most of the children, but the exception of Marie reminds us that webbing is in the end a personal experience. For her, the cards played a much more significant role.

The webbing in the later phases had a different pattern: (map, number)→number. The role of the map was important. Even then this is an oversimplification. This schematic fails to recognise the importance of other internal resources, which were either reinforced or occasionally denied during the activity. The construct of situated abstraction informs the webbing process; we recognised the use of geographically-oriented situated abstractions, some of which were obscure in origin.

Without doubt the drive for this webbing activity came from a perception of the task as engaging and purposeful, in the sense of Ainley and Pratt (2002). The children could relate to the task at an age where Father Christmas still holds some magic. However, this engagement is insufficient. The task was intended as one that would bring together through use the geographical concept of latitude and the mathematical concept or skill of interpolating directed number. We observed through the webbing activity evidence of the construction of two utilities, the first for directed number and the second for latitude:

1. Directed numbers *inter alia* were used to locate countries.
2. Knowledge of the relative position of countries on the map *inter alia* was used to interpolate and extrapolate directed numbers.

Father Christmas's epic journey was successfully designed in order that the purpose of the task led almost inevitably to the construction of these two utilities. We believe that by paying attention to purpose and utility in designing sense-making tasks, teachers can avoid the McDonald's approach and provide their children with a rich and stimulating diet.

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[1]The Father Christmas character contains a small RFID tag, which activates information about the country. This study is part of a European project, called Webkit: Intuitive physical interfaces to the WWW (IST-2001-34171), which is exploring the use of Tangible User Interfaces (TUIs) in education.